Metric Spaces and Topology
Lecture

Prop. For compactness, it's sufficient to consider open covers consisting of basic open sets (i.e. sets in some fixed basis).
More precisely, a space X is compact if and only if for any/some basis $\mathcal{B}$ of X, any open cover of X consisting of sets in $\mathcal{B}$ has a finite subcover.
Similarly, it's enough to consider complements of basic open sets in the equiv. def. of compactness with closed sets.

Proof. Given an open cover $\mathcal{U}$, we take its refinement $\mathcal{U}' := \{B \in \mathcal{B} : \exists U \in \mathcal{U} \text{ s.t. } B \subseteq U\}$. $\mathcal{U}'$ is also a cover

of X because every $U \in \mathcal{U}$ is a union of $B \in \mathcal{B}$ s.t. $B \in \mathcal{U}$. If $\mathcal{U}'$ has a finite subcover $B_1, B_2, \dots, B_n$, then so does $\mathcal{U}$, indeed, for each B_i $\exists\, U_i \in \mathcal{U}$ s.t. $B_i \subseteq U_i$, so $U_1, U_2, \dots, U_n$ is also a cover of X. $\square$

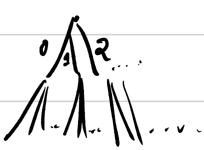
Examples/nonexamples. (a) Finite topologies are compact, i.e. those topol. spaces which have only finitely many open sets, e.g. whose underlying set of points is finite, or the trivial topology.

(b) $\mathbb{R}$ is not compact. Indeed, $\mathbb{R} = \bigcup_{n \in \mathbb{N}} (-n, n)$, but the cover $\{(-n,n) : n \in \mathbb{N}\}$ doesn't have a finite subcover.
In fact:

Prop. If a top space admits an unbdd compatible metric, then it's not compact.

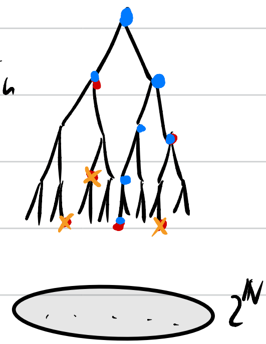
Proof. If d is an unbdd metric on X and $x_0 \in X$, then $\{B_n(x_0) : n \in \mathbb{N}\}$ is an open cover, which doesn't have a finite subcover. $\square$

(c) $\mathbb{N}^{\mathbb{N}}$ is noncompact. (Note that the usual metric on $\mathbb{N}^{\mathbb{N}}$ is bdd by 1, so the unbddness isn't necessary for noncompactness of a metric space.)
Indeed, $\mathbb{N}^{\mathbb{N}} = \bigcup_{n \in \mathbb{N}} [n]$, so $\{[n]\}_{n \in \mathbb{N}}$ is an open cover of $\mathbb{N}^{\mathbb{N}}$ with ∞ many disjoint nonempty open sets, hence $\nexists$ finite subcover.



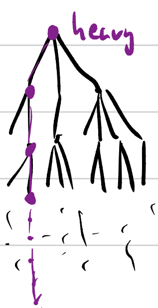
(d) $2^{\mathbb{N}}$ is compact.

Proof 1. Let $\mathcal{U}$ be an open cover of $2^{\mathbb{N}}$ with cylindrical sets $[w]$, $w \in 2^{<\mathbb{N}}$. We may assume without loss of generality that the sets in $\mathcal{U}$ are maximal, i.e. if $[w] \in \mathcal{U}$ and $[w'] \supsetneq [w]$, then $[w'] \notin \mathcal{U}$. We claim that $\mathcal{U}$ is finite. To prove this, we form a tree whose leaves are the words w s.t. $[w] \in \mathcal{U}$, namely:



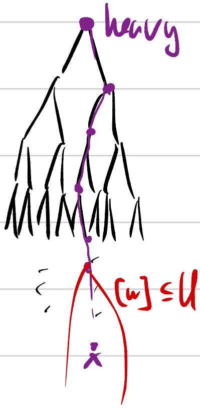
$$T := \{ w \in 2^{<\mathbb{N}} : \exists [w'] \in \mathcal{U} \text{ s.t. } w \subseteq w' \}.$$

König's lemma. Every finitely branching infinite tree has an infinite branch.

Proof. 

Call a vertex w in T heavy if the tree below it is infinite. By our hypothesis, the root of T is heavy. Then at least one of its finitely many children must be heavy, choose one. Then again one of its children must be heavy (by the pigeonhole principle). And so on, we obtain an infinite branch.

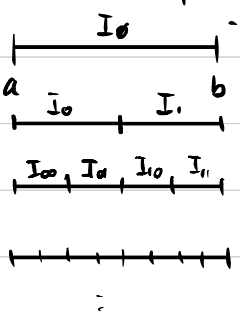
By König's lemma our T must be finite since otherwise it has an infinite branch x, i.e. $\forall n\ x|_n \in T$, so $\nexists [w] \in \mathcal{U}$ s.t. $x \in [w]$, contradicting $\mathcal{U}$ being a cover of $2^{\mathbb{N}}$. $\square$



Proof 2. Let $\mathcal{U}$ be an open cover of $2^{\mathbb{N}}$ and suppose it doesn't have a finite subcover. Call a vertex $w \in 2^{<\mathbb{N}}$ heavy if $[w]$ cannot be covered by a finite subset of $\mathcal{U}$. Thus $\emptyset \in 2^{<\mathbb{N}}$ is heavy, and hence one of its children must be heavy, ... we obtain an infinite branch $x \in 2^{\mathbb{N}}$ s.t. $\forall n\ x|_n$ is heavy. But $\exists\, U \in \mathcal{U}$ s.t. $x \in U$ so for some n, $[x|_n] \subseteq U$, hence $x|_n$ isn't heavy. $\square$

(e) The intervals $[a,b] \subseteq \mathbb{R}$ are compact.

Proof.



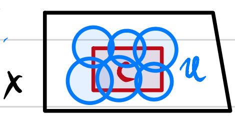
Let $I_\emptyset := [a,b]$, then let I_0 and I_1 be the first and second closed half intervals. We repeat and get $I_{00}, I_{01}, I_{10}, I_{11}$. Thus we obtain a sequence $(I_s)_{s \in 2^{<\mathbb{N}}}$ of closed intervals s.t. $|I_s| = \frac{b-a}{2^{|s|}}$.

Let $\mathcal{U}$ be an open cover of $I_\emptyset$. Call I_s, $s \in 2^{<\mathbb{N}}$, heavy if $\nexists$ finite subset of $\mathcal{U}$ that covers I_s.
The rest is similar to Proof 2 of $2^{\mathbb{N}}$ being compact, but here we need to use the nested intervals lemma. $\boxed{HW}$ $\square$

Remark on open covers of subspaces. For top. space X, when considering whether a subspace $Y \subseteq X$ is compact (in the relative top.), we have to consider covers $\mathcal{U}$ of Y with relatively open subsets $U \subseteq Y$. But for each $U \in \mathcal{U}$, there is an open set $\tilde{U} \subseteq X$ s.t. $U = \tilde{U} \cap Y$, so the collection $\tilde{\mathcal{U}} := \{\tilde{U} \subseteq X : \tilde{U} \text{ is open in } X \text{ and } \tilde{U} \cap Y \in \mathcal{U}\}$ is still a cover of Y in the sense that $Y \subseteq \bigcup \tilde{\mathcal{U}}$, but it consists of sets that are open in X. It remains to note that $\mathcal{U}$ has a finite subset covering Y if and only if $\tilde{\mathcal{U}}$ does.

Prop. Closed subsets of compact spaces are compact.

Proof.



Let $Y \subseteq X$ be a closed subset of a compact X. Let $\mathcal{U}$ be a cover of Y with open subsets of X (see the remark above). Then $\mathcal{U} \cup \{Y^c\}$ is an open cover of X, so so $\exists$ finite subcover $U_1, U_2, \dots, U_n \in \mathcal{U} \cup \{Y^c\}$. Only one of the U_i is Y^c, and removing it we still get a finite cover of Y with sets in $\mathcal{U}$. $\square$

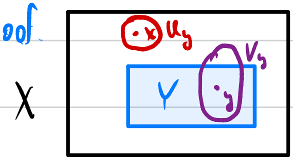
Counter-examples to converse. (a) In the half-open space, i.e. $X := \{0,1\}$ and $\mathcal{T} := \{X, \emptyset, \{1\}\}$, the set $Y := \{1\}$ is compact not closed.

(b) In $X := \mathbb{R}$ with the cofinite top (= Zariski top), every subset is compact $\boxed{\text{HW}}$ but only finite subsets are closed.

The last example shows that T_1 is not enough for compact subsets to be closed, however T_2 is enough:

Prop. In Hausdorff top. spaces, compact subsets are closed.

Proof.



Fix $x \in X \setminus Y$ & try to find an open neighb. $U \ni x$ disjoint from Y. For any $y \in Y$, there are open disjoint $U_y \ni x$ & $V_y \ni y$. Then $\{V_y : y \in Y\}$ is an open cover of Y so it has a finite cover $V_{y_1}, V_{y_2}, \dots, V_{y_n}$. Then $U := \bigcap_{i=1}^{n} U_{y_i}$ is still an open neighb. of x disjoint from $\bigcup_{i=1}^{n} V_{y_i} \supseteq Y$. $\square$

Example. Non-closed sets of metric spaces are not compact.